RATIONAL CURVES ON HYPERSURFACES OF A PROJECTIVE VARIETY

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OCT 19, 2014

Abstract. In this paper, we extend our result in [3] to hypersurfaces of any smooth projective variety Y. Precisely we let X_0 be a generic hypersurface of Y and $c_0 : \mathbf{P}^1 \to X_0$ be a generic birational morphism to its image, i.e. $c_0 \in Hom_{bir}(\mathbf{P}^1, X_0)$ is generic, such that

(1) $dim(X_0) \geq 3$,

(2) $H^1(N_{c_0/Y}) = 0$.

Then

$$H^1(N_{c_0/X_0}) = 0. \tag{0.1}$$

As an application we prove that the Clemens' conjecture holds for Calabi-Yau complete intersections of dimension 3.

1 Statement

We work over complex numbers, $\mathbb{C}$. A general or generic hypersurface is referred to a hypersurface as a point in a complement of a countable union of proper closed subsets of the space of all hypersurfaces. Let Y be a smooth projective variety with $dim(Y) \geq 4$. Let $\mathcal{L}$ be a very ample line bundle on Y such that $dim(H^0(\mathcal{L})) \geq 5$. Let

$$X_0 = div(f_0) \subset Y$$

where $f_0 \in H^0(\mathcal{L}^h)$ is generic. Let

$$c_0 : \mathbf{P}^1 \to C_0 \subset X_0 \subset Y$$

be a birational morphism onto an irreducible rational curve C_0. Because the induced sheaf morphism

$$(c_0)_s : \mathcal{O}_Y \quad \to \quad \mathcal{O}_{\mathbf{P}^1} \tag{1.1}$$

is surjective. We have well-defined normal sheaves N_{c_0/X_0} and $N_{c_0/Y}$ both on $\mathbf{P}^1$.

Theorem 1.1. *(Main Theorem). Let f_0 be generic. If c_0 is generic in an irreducible component of $Hom_{bir}(\mathbf{P}^1, X_0)$, and*

$$H^1(N_{c_0/Y}) = 0, \tag{1.2}$$

then

$$H^1(N_{c_0/X_0}) = 0. \tag{1.3}$$

2 Application

Our first application of the result is the proof of Clemens' conjecture for Calabi-Yau complete intersections of dimension 3. More precisely we call the following statement of the corollary, Clemens' conjecture for Calabi-Yau complete intersections.

Corollary 2.1. *Let X_0 be a generic, Calabi-Yau complete intersection of dimension 3. Let $c_0 : \mathbf{P}^1 \to X_0$ be a morphism that is birational to its image. Then*
(1) c_0 is an immersion,
(2) and N_{c_0/X_0} is a vector bundle satisfying

$$N_{c_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus O_{\mathbf{P}^1}(-1). \tag{2.1}$$

Remark
The name "Clemens' conjecture for complete intersections" is known for the following reason. It follows from the corollary that if they exist, there are finitely many rational curves C_0 of each degree on X_0 such that they are all immersed with normal bundle (2.1) for its normalization c_0. The corollary does not prove the existence of rational curves. All these assertions for quintic threefolds X_0 are main parts of Clemens' conjecture.

Proof. (following the main theorem). Let's assume $X_0 \subset \mathbf{P}^n$ and X_0 is defined by the polynomials $f_1, \cdots, f_{n-3}$. Let $deg(f_i) = h_i$. Let

$$Y_i = \cap_{j=1}^{j=i} \{f_j = 0\}. \tag{2.2}$$

Let S be the space of the complete intersections containing X_0. So $Y = Y_{n-3}$.

Then applying main theorem inductively on Y_i, we obtain that for generic $c_0 \in Hom_{bir}(\mathbf{P}^1, X_0)$, the normal sheaf N_{c_0/X_0} exists and

$$H^1(N_{c_0/X_0}) = 0. \tag{2.3}$$

Next we show c_0 is an immersion. Let Γ be an irreducible component of the incidence scheme

$$\{(c, [f]) \in M \times S : f(c) = 0\} \tag{2.4}$$

that dominates S, where $M = \oplus^{n+1} H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ for some positive integer d. Let Γ_{X_0} be the fibre of Γ over the generic X_0. Then as in lemma 2.8, [3], for a generic $c_0 \in \Gamma_{X_0}$,

$$\frac{T_{c_0}\Gamma_{X_0}}{ker} \simeq H^0(c_0^*(T_{X_0})), \tag{2.5}$$

where ker is the line in M going through $c_0 \neq 0$ and 0. By (2.3),

$$dim(T_{c_0}\Gamma_{X_0}) = 4. \tag{2.6}$$

Now we consider it from a different point of view. Because c_0 is a birational map to its image, there are finitely many points $t_i \in \mathbf{P}^1$ where the differential map

$$(c_0)_* : T_{t_i}\mathbf{P}^1 \quad \to \quad T_{c_0(t_i)}X_0 \tag{2.7}$$

is not injective. Assume its vanishing order at t_i is m_i . Let

$$m = \sum_i m_i. \tag{2.8}$$

Let $s(t) \in H^0(\mathcal{O}_{\mathbf{P}^1}(m))$ such that

$$div(s(t)) = \Sigma_i m_i t_i.$$

The sheaf morphism $(c_0)_*$ is injective and induces a composition morphism ξ_s of sheaves

$$T_{\mathbf{P}^1} \stackrel{(c_0)_*}{\rightarrow} c_0^*(T_{X_0}) \otimes \mathcal{I}_{div(s)} \stackrel{\frac{1}{s(t)}}{\rightarrow} c_0^*(T_{X_0}) \otimes \mathcal{O}_{\mathbf{P}^1}(-m), \tag{2.9}$$

where $\mathcal{I}_{div(s)}$ is the ideal sheaf of $div(s)$. It is easy to see that the induced bundle morphism ξ_b is injective. Let

$$N_m = \frac{c_0^*(T_{X_0}) \otimes \mathcal{O}_{\mathbf{P}^1}(-m)}{\xi_b(T_{\mathbf{P}^1})}. \tag{2.10}$$

Then

$$dim(H^0(N_m)) = dim(H^0(c_0^*(T_{X_0}) \otimes \mathcal{O}_{\mathbf{P}^1}(-m))) - 3. \tag{2.11}$$

On the other hand, three dimensional automorphism group of $\mathbf{P}^1$ gives a rise to a 3-dimensional subspace Au of

$$H^0(c_0^*(T_{X_0})).$$

By (2.6), $Au = H^0(c_0^*(T_{X_0}))$. Over each point $t \in \mathbf{P}^1$, Au spans a one dimensional subspace. Hence

$$c_0^*(T_{X_0}) = \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbf{P}^1}(-k_2), \tag{2.12}$$

where k_1, k_2 are some positive integers. This implies that

$$dim(H^0(c_0^*(T_{X_0}) \otimes \mathcal{O}_{\mathbf{P}^1}(-m))) = dim(H^0(\mathcal{O}_{\mathbf{P}^1}(2-m)). \tag{2.13}$$

Then

$$dim(H^0(c_0^*(T_{X_0}) \otimes \mathcal{O}_{\mathbf{P}^1}(-m))) = 3 - m. \tag{2.14}$$

Since $dim(H^0(N_m)) \geq 0$, by (2.11), $-m \geq 0$. By the definition of m, $m = 0$. Hence c_0 is an immersion.

Next we prove (2). Notice that $(c_0)_*(T_{\mathbf{P}^1})$ is a subbundle generated by global sections. It must be the $\mathbf{O}_{\mathbf{P}^1}(2)$ summand in (2.12) because k_1, k_2 are positive. Therefore

$$N_{c/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(-k_1) \oplus \mathcal{O}_{\mathbf{P}^1}(-k_2). \tag{2.15}$$

Since $deg(c_0^*(T_{X_0})) = 0$, $k_1 = k_2 = 1$. Part (2) is proved.
□

Further applications can be expected. In the light of theorem 1.1, the Calabi-Yau condition is not responsible for the vanishing of the first cohomology. Let's be more specific. The main theorem states the condition

$$H^1(N_{c_0/X_0}) = 0$$

is descending to hypersurfaces as the dimension of the complete intersection is descending. Such descending will not stop until the dimension reaches 3. Therefore the vanishing property

$$H^1(N_{c_0/X_0}) = 0$$

holds for all generic complete intersections whose dimensions are larger than or equal to 3. The vanishing of the first cohomology of the normal bundle is originated from the projective space. Its disconnection with Calabi-Yau condition gives a wide range of applications. Those applications include the extensions of results in [1] to general Fano complete intersections, and extensions of related results and conjectures in [2] to general complete intersections of corresponding types. We will discuss these extensions separately elsewhere

3 Proof of the main theorem

By the assumption of the main theorem, we may regard

$$Y \subset \mathbf{P}^n \tag{3.1}$$

as a smooth subvariety of dimension $e \geq 4$, and $\mathcal{L} = \mathcal{O}_{\mathbf{P}^n}(1)|_Y$. Let

$$f_0 \in H^0(\mathcal{L}^h) \tag{3.2}$$

be generic such that

$$div(f_0) = X_0. \tag{3.3}$$

Next we choose subspaces $V_1 \simeq \mathbf{P}^{n-e-1}$, $V_2 \simeq \mathbf{P}^e$ of $\mathbf{P}^n$ such that

$$V_1 \cap Y = \varnothing.$$

The projection Pr

$$\mathbf{P}^n \backslash V_1 \quad \stackrel{Pr}{\rightarrow} \quad V_2 \tag{3.4}$$

is restricted to a generically finite-to-one map from Y. The projection $Pr(X_0)$ is a generic hypersurface of V_2 because f_0 is generic. We denote the section that defines $Pr(X_0)$ by $Pr(f_0)$. Now we consider the rational curves. Notice that Pr induces a rational map

$$\begin{array}{cccc} \eta : \Gamma_{X_0} & \dashrightarrow & \Gamma_{Pr(f_0)} \\ c_0 & \rightarrow & Pr(c_0) = Pr \circ c_0. \end{array} \tag{3.5}$$

We may assume that c_0 lies in the regular locus of η, and $\eta(c_0)$ is not a constant map for generic $c_0 \in \Gamma_{X_0}$. The map is clearly surjective. Then if $Pr(c_0) \in \Gamma_{Pr(f_0)}$ is generic, there is an inverse $c_0 = \eta^{-1}(c_0)$ in Γ_{X_0} which is also generic in $Hom(\mathbf{P}^1, X_0)$.

Let $U \subset \mathbf{P}^1$ be the open set of $\mathbf{P}^1$ such that the $Pr|_Y$ at $c_0(U)$ is smooth. Let $c_0', Pr(c_0)'$ be the restriction of $c_0, Pr(c_0)$ to U. Then we have commutative diagram

$$\begin{array}{ccc} N_{c_0'/Y} & \rightarrow & N_{Pr(c_0)'/V_2} \\ \downarrow & & \downarrow \\ (c_0')^*(N_{X_0/Y}) & \rightarrow & (Pr(c_0)')^*(N_{Pr(X_0)/V_2}). \end{array} \tag{3.6}$$

Now we apply theorem 1.1 in [3] to the generic hypersurface $Pr(X_0)$ of the projective space V_2. We obatin that $H^1(N_{Pr(c_0)/X_0}) = 0$. By the exact seqence

$$H^0(N_{Pr(c_0)/V_2}) \quad \to \quad H^0(Pr(c_0)^*(N_{Pr(X_0)/V_2})) \quad \to \quad H^1(N_{Pr(c_0)/X_0}) = 0,$$

we obtain that the map

$$H^0(N_{Pr(c_0)/V_2}) \quad \to \quad H^0(Pr(c_0)^*(N_{Pr(X_0)/V_2})) \tag{3.7}$$

is surjective. Applying this surjectivity to the diagram (3.6), we obtain that the natural map

$$H^0(N_{c_0/Y}) \quad \stackrel{\pi_1}{\to} \quad H^0((c_0)^* N_{X_0/Y}) \tag{3.8}$$

is also surjective. Considering the exact sequence

$$0 \quad \to \quad N_{c_0/X_0} \quad \to \quad N_{c_0/Y} \quad \to \quad c_0^*(N_{X_0/Y}) \quad \to \quad 0, \tag{3.9}$$

we obtain another exact sequence

$$H^0(N_{c_0/Y}) \quad \stackrel{\pi_1}{\to} \quad H^0(c_0^*(N_{X_0/Y}) \quad \stackrel{\pi_2}{\to} \quad H^1(N_{c_0/X_0}) \quad \to \quad H^1(N_{c_0/Y}) = 0.$$

Since π_1 is surjective, image of π_2 must be 0. Because of the assumption, $H^1(N_{c_0/Y}) = 0$, $H^1(N_{c_0/X_0})$ must be zero. The main theorem is proved.

REFERENCES

[1] J. Harris, M. Roth and J. Star, *Rational curves on Hypersurfaces of low degree*, J. Reine Angew. Math., 571 (2004), pp. 45-96.
[2] C. Voisin , *On some problems of Kobayashi and Lang: Algebraic approaches*, International Press, Current developments in mathematics (2003), pp. 53–125.
[3] B. Wang, *Rational curves on hypersurfaces*, ArXiv 1410.3090, 2014